A GAMING ANALYSIS OF COUNTER-INFILTRATION OPERATIONS

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THESIS

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ABSTRACT

An analysis is made of the allocation problem associated with the conduct of ambush operations to interdict infiltration routes in a guerrilla-counterguerrilla environment. A multi-stage two-person non-zero sum game is used to model that allocation problem. It is shown that Lanchester's equations can be used to develop a criterion function, related to the casualty ratio, which demonstrates the minimax property. The game is then solved to determine the optimal allocations for both the guerrilla and the counterguerrilla and the value of the game for two different forms of the criterion function. The two results are compared and the usefulness of the casualty ratio as a measure of effectiveness is discussed.

TABLE OF CONTENTS

SUMMARY

This study addresses an allocation problem which occurs frequently in the conduct of modern counterguerrilla campaigns. These campaigns often include counter-infiltration operations aimed at disrupting or destroying the lines of communication between the guerrillas within the embattled area and their political or military supporters in adjacent areas. Often these operations are the responsibility of the local military commanders of the areas through which the infiltration is being effected, with those commanders being limited as to the number and type of forces they can commit. This thesis is an application of the mathematical theory of games to determine the optimal allocation of ground forces for the conduct of ambush interdiction operations.

Section II is a general description of the infiltration situation and a preliminary formulation of the problem as a multi-stage non-zero sum game. The infiltration network is idealized as a flow network and the operations of both the insurgent and the counter-insurgent are studied for only the arcs in the minimum cut set of the network which are called infiltration routes. Each of these routes constitutes one stage of a multi-stage game. At each stage of the game the opponents each commit some portion of their total resources to the route in question, receiving some return which is a function of the commitments of both. The

overall return for a play of the game is therefore determined in part by the return from each stage, and the principle of optimality is used to decompose the multistage game into a sequence of single-stage games.

In Section III, Lanchester's equations of combat are used to determine the combat outcome at each stage in terms of the casualties sustained by the opposing forces. These casualty functions are then used to define a criterion function which is shown to possess the minimax property. In addition, the stage transformations are derived for both players. These functions are then used in Section IV to formulate the game in its final form. After showing that the method of decomposition is justifiable by demonstrating the sufficient conditions, the game is solved. It is shown that the optimal strategy for the insurgent, if he wishes to minimize the criterion or casualty ratio, is to commit no forces to the infiltration network at all, and the optimal strategy for the counter-insurgent is to play any possible strategy.

Section V is an extension of the original model to more nearly account for the insurgent's true goal of maximizing the forces successfully infiltrated through the network. A new individual return for the insurgent is used to define a new criterion function. This criterion is then used to solve the game for the special case in which the initial forces of the opposing players are strategically or Lanchester equivalent. It is shown that the optimal strategy for both players is to commit their entire initial forces to a randomly selected route, choosing each route with equal probability. It is also shown that this strategy virtually insures that the insurgent will successfully infiltrate a major portion of his force.

The discussion in Section VI is concerned primarily with the applicability of the assumptions made in reducing the allocation problem to a game of strategy. The results of the two formulations suggest that the casualty ratio is not a true measure of the effectiveness of the counterinsurgent in limiting the infiltration through the network. In addition, the possible extension of the general method to situations with a number of dissimilar operational alternatives should be of interest, along with comparable extensions to games with positive critical survivor levels and games of partial information.

Included as Appendix A is the formulation and proof of an extended theorem of continuous games which insures that the criterion function used in the paper demonstrates the minimax property. The theorem could also be of use in the analysis of other similar problems where the criterion is reasonable.

I. INTRODUCTION

A major objective in the conduct of military operations in an insurgent environment is that of limiting the facility with which the insurgents maintain the lines of communication between their sanctuaries and their areas of operation. This paper describes an application of the mathematical theory of games to the study of the allocation problem involved in mounting ambush interdiction operations along guerrilla infiltration routes. Before formulating the model, the rationale behind the approach used will be explained and the aim of the study will be detailed.

A. THE COUNTER-INFILTRATION PROBLEM

Interdiction operations are designed to disrupt or curtail the enemy's use of its lines of communication and supply. Although historically of secondary importance, interdiction and harrassment are distinctive characteristics of unconventional or guerrilla warfare. With the increasing number of guerrilla conflicts of the last two decades these tactics have become especially important since they have required the expenditure of a disproportionate part of the total military effort involved in conducting successful counter-insurgency campaigns.

The insurgent maintains the initiative in most guerrilla conflicts. Organized in small, highly mobile groups, the insurgent unit can choose to strike only when it has overwhelming, though momentary, superiority of force. When outnumbered or under pressure the same unit usually disbands to disguise itself among the normal populace of its area of operations. To maintain any significant level of activity, however, the unit must receive logistical and command support from its parent organization.

The insurgent logistical base and the target area of operations are seldom contiguous. The former is ideally located in some area or sanctuary which is inaccessible to the counter-insurgent due to political or physical limitations, while the latter is within the political sphere of the counter-insurgent. These two areas are usually separated by a border area characterized by extremely difficult terrain and sparse population. Though of little tactical interest, this area is strategically important to both antagonists. The insurgent relies on the use of this area for the infiltration of men and materiel and the exfiltration of casualties and intelligence in support of operations in his target area. The relative ease with which these infiltration operations can be exposed and interdicted make the same area a prime target for the counter-insurgent operations.

Military experience in Malaysia, Algeria and Vietnam has shown that the insurgent traffic tends to be concentrated on a small number of routes through these border areas [12, 14]. This is a result of the difficulty of the terrain, which limits the flow capacity of unimproved routes, and of the limited resources and communications capability of the insurgent. In Malaysia the communist terrorists consistently used a few well-defined routes even in an open jungle which offered no limitations to movement, seemingly out of habit [8]. This channeling of the insurgent logistical effort increases the effectiveness of properly conducted interdiction operations in this border area.

Although interdiction alone seldom eliminates the insurgent threat it has been a principal factor in the conduct of all recent successful counter-insurgent campaigns [11]. Effective interdiction imposes on the insurgent an immediate loss in personnel and supplies. More important, the operational effectiveness of the insurgent force in the target area of operations is lowered, reducing the level of conflict and increasing the vulnerability of the insurgent in this area. Finally, if completely sealed off from its support due to total interdiction of the lines of communication, the insurgent force must often cease offensive operations and concentrate on survival.

The interdiction tactics which have been employed in these situations have been varied, including long-range artillery bombardment, air interdiction, and ground operations of varied types. One of the most effective tactics has been the use of ambushes along the supply routes. The very channeling noted above makes these routes extremely vulnerable to attack by wel organized and well-armed ambushing forces. Furthermore, the tactical advantage provided by the planned ambush multiplies the effective size of the ambushing force, enabling it to achieve results far out of proportion to the actual resources employed. The importance of the ambush is recognized in the operational doctrine of most modern insurgent and counter-insurgent forces, for the same factors which make the ambush a tactic of choice for the insurgent make it an effective means of combatting him.

The allocation problem involved in conducting ambush operations to interdict guerrilla infiltration routes is one of allocating the military resources available among the possible ambush sites or supply routes. The lack of any formal planning model for the conduct of these operations and their importance in current military planning make the problem a valid subject of study.

B. OBJECTIVE AND SCOPE

The object of this paper is to develop a method of optimally allocating ground forces for the conduct of ambush operations. The problem will first be formulated as a two-person, multi-stage game. An attempt will then be made to determine a realistic payoff function for the game, and the applicability of the zero-sum assumption will be examined. The solution to the game will be determined to yield the value of the game and the optimal strategies for both antagonists. The sensitivity of the solution to some of the assumptions will be investigated and some extensions of the method of approach proposed.

To facilitate the solution of the problem the formulation and proof of an extended min-max theorem for a certain class of continuous games is necessary. The complete statement and proof of the theorem will be included in Appendix A along with a brief commentary on its possible future applications.

II. THE ALLOCATION PROBLEM AS A GAME

A military commander with tactical responsibility for an area through which infiltration is being effected faces the problem of allocating his resources for interdiction to maximize their effect. The nature of the allocation process and the difficulties in analyzing it will be studied in detail, and a simple introduction to the idea of a game of strategy will be made. A general formulation of the allocation process as a game will be followed by an explanation and justification of the assumptions made in the formulation. The concept of a return or payoff will be introduced in the general sense with the actual specification of the function being made in a later chapter.

A. THE NATURE OF THE PROBLEM

The allocation process to be discussed differs from that of most current allocation models such as those formulated by Bellman [2] and Danskin [4]. Consider the problem faced by the counter-insurgent. His task is that of simultaneously allocating some limited resources among a number of feasible alternatives in an attempt to maximize some return. This return is a function not only of the resources he commits to the different ventures, but also of the resources committed by his opponent, the insurgent, to counter these operations. The insurgent is also attempting to maximize some return which may not be related to that of the counter-insurgent in a manner which facilitates mathematical analysis. The resources involved are usually military forces or hardware, and the return is some measure of the harm done to the opponent or his cause.

The major differences between this allocation process and some of the current allocation models lie in the assumptions made about the time frame and information transmission involved. Danskin's model [4] is representative in that it assumes that the actions of the two antagonists are sequential and distinct and that the second player has perfect knowledge of the play made by his opponent and responds accordingly. However, in the situation currently studied these assumptions are seldom valid. Both the insurgent and the counter-insurgent must mount operations in the border area on a day-to-day basis. The two allocations are simultaneous in nature since the time lapse between them is usually too short to permit either combatant to take advantage of any knowledge obtained about the other's current allocation. Furthermore, such information is seldom current, at best imperfect and can only be used to determine a very rough <u>a priori</u> distribution on the possible allocations of the opponent.

The major similarity between the ambush allocation problem and the models mentioned above is that the military allocation problem has all of the aspects of a game of strategy since there are two opponents with divergent aims, each with the ability to determine in part the return of his adversary. A brief introduction to the theory of games will be used to introduce the notation employed in the formulation.

B. GAMES OF STRATEGY

The theory of games of strategy is a mathematical theory of competitive decision-making. Games of strategy and games of pure chance differ in that only in the former can the participant bring any influence to bear on the outcome of an event. As a result, intelligence and skill can be useful in the play of games of strategy.

As it is used in this report, the word <u>game</u> will denote a set of rules for conducting a decision process, while the word <u>play</u> will refer to a particular realization of the process. The participants of the game will be called <u>players</u>, and the points in a game at which one of the players picks an alternative action will be referred to as <u>stages</u> while the actual alternative selected will be called a <u>choice</u>. The payment which the players receive from a play of the game will be called the <u>return</u> or <u>payoff</u>.

Games can be either single-stage or multi-stage games, with a multi-stage game being one in which there is more than one choice made by the players in a single play of the game. Games are also classified according to certain characteristics, such as the number of players and the form of payoff. The class of interest will be the class of two-person, infinite games. There are two players in a game of this class, and the choices of the players are made from infinite sets. The result of a play of the game is a payoff which is a function of the choices made by both players. When the choices are limited to the closed interval [0, 1], the game is said to be continuous. In continuous games the players choose from [0, 1] by means of probability distribution functions which are called <u>strategies</u>. A game is said to be <u>zero sum</u> if there is no change in the total resources in the player system during a play of the game, that is, when the payoff to any player is made by another player in the game.

For a two-person zero sum continuous game, let the payoff be M and let one player choose a number x from [0, 1] by means of a distribution function F. Let the other player choose a number y from [0, 1] by means of a distribution function G. The <u>expected payoff</u> to the first player is defined as

$$E(F,G) = \int_0^1 \int_0^1 M(x,y)\, dF(x)\, dG(y).$$

If, for a continuous two-person zero sum game, the values

$$v_1 = \operatorname*{Max}_F \operatorname*{Min}_G E(F,G)$$

and

$$v_2 = \underset{G}{\text{Min}}\ \underset{F}{\text{Max}}\ E(F,G)$$

both exist and are equal, their common value is called the <u>value</u> of the game, and the distribution functions F_o and G_o which yield this common value are called the <u>optimal strategies</u> for the respective players. A game is said to be solved when the value of the game and the optimal strategies have been determined [7].

C. FORMULATION OF THE GAME

In the formulation which follows the terminology developed above will be used freely. Notation will be introduced and defined as it is needed.

1. <u>Scenario</u>

The infiltration routes through a border area can be represented by a network as shown in Figure 1. The insurgent attempts to move men and supplies from the sanctuary area through the network into the area of operations, or in the reverse direction. The counter-insurgent attempts to stop this flow by mounting ambushes along the arcs of the network. It will be assumed that both the insurgent and the counter-insurgent have complete knowledge of the entire network, and that all of the arcs in the network are equally accessible to both.

In any network there is at least one set of arcs whose removal completely separates the source or origin of flow

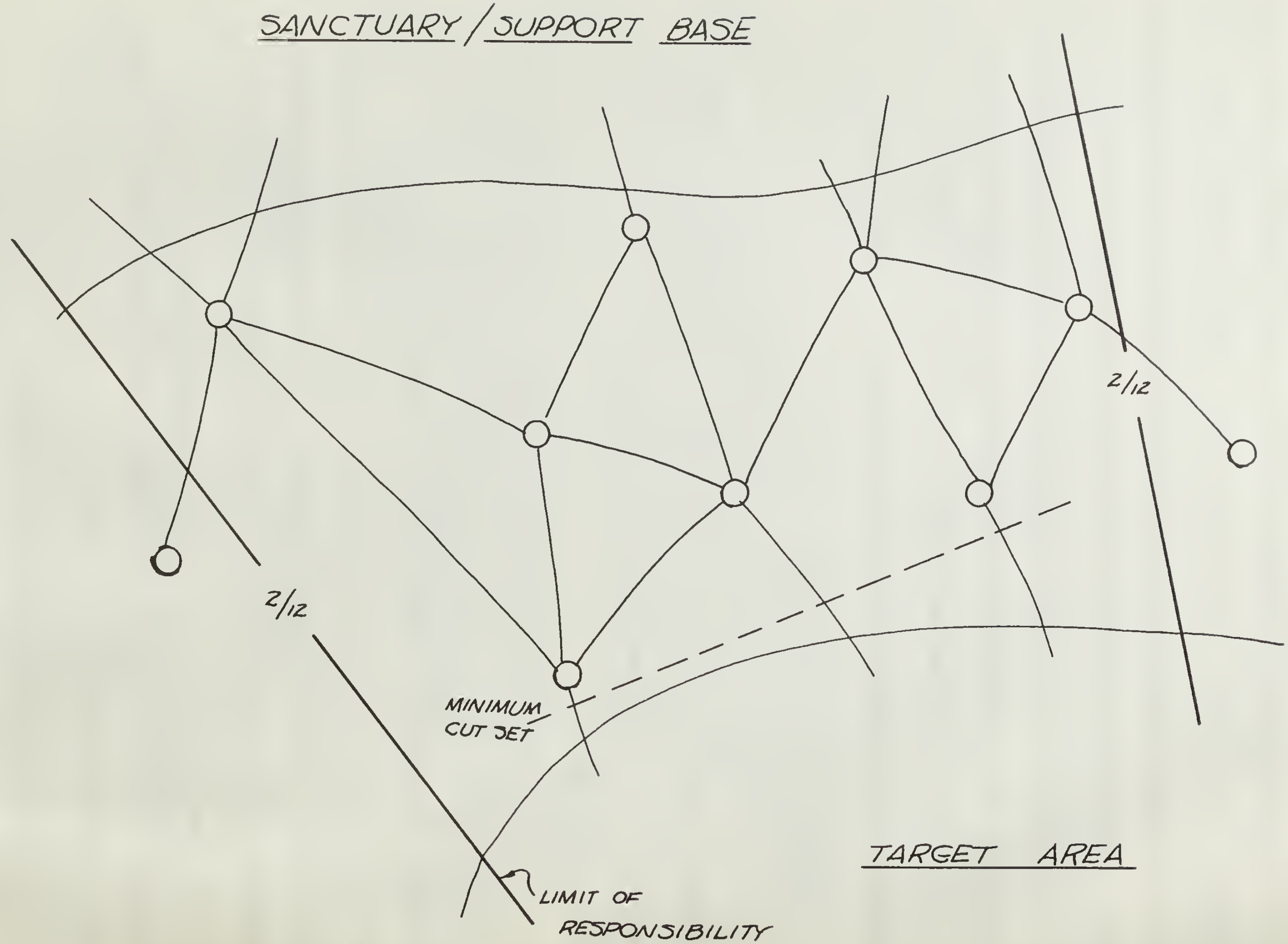

Figure 1. Network Representation of Infiltration Network

from the sink or destination. Such a set of arcs is called a cut set, and the cut set with the fewest number of arcs of all the cut sets of a given network is called the minimum cut set. By operating along all of the arcs of a cut set the counter-insurgent can control the flow through the network, and by operating along the minimum cut set he can control the flow with the least expenditure of his resources, if it is assumed that all arcs are equally accessible and suitable for the ambush operations. For allocation purposes then, the entire network can be replaced by a network consisting only of simple arcs from the sanctuary area to the destination or target area, with the number of arcs in the new network being equal to the number of arcs in the minimum cut set of the original network.

For the present model the arcs in the modified network will be called infiltration routes. It will be assumed that there is no limit on the flow along these routes and that they are all equally capable of being ambushed successfully by the counter-insurgent force. The allocation of the forces of both the insurgent and the counter-insurgent to the different infiltration routes will now be formulated as a game.

2. General Formulation

The allocation game will be described in the following manner. Let the counter-insurgent commander be Player A, and let the insurgent be Player B, and let the initial resource constraint for the two players be A and B respectively. Let the number of infiltration routes through the border area be n. At each play of the game Player A chooses a vector.

$$f = (f_1, f_2. \ldots, f_i, \ldots, f_n)$$

from a set Δ, and Player B chooses a vector

$$g = (g_1, g_2, \ldots, g_i, \ldots, g_n)$$

from a set Γ, where both Δ and Γ lie in Euclidean n-space and where f_i and g_i are the forces allocated by the respective players to the i^{th} route. The sets described are,

$$\Delta = \{f \mid \sum_{i=1}^{n} f_i \leq A, \; f_i \geq 0\}$$

$$\Gamma = \{g \mid \sum_{i=1}^{n} g_i \leq B, \; g_i \geq 0\}.$$

As a result of the allocation Player A receives a payoff of R(f,g; A,B) and Player B a payoff of R'(f,g; A,B). To simplify notation, R(f,g; A,B) will be written as R(f,g). For the present it will be assumed that R(f,g) is continuous over Δ and Γ and that

$$R(f,g) + R'(f,g) = 0$$

that is, that the process is zero sum. The value of the game described is given by the expression

$$v_n = \underset{F'}{\mathrm{Max}}\ \underset{G'}{\mathrm{Min}} \iint R(f,d)\ d\,F'(f)\ d\,G'(g)$$

$$= \underset{G'}{\mathrm{Min}}\ \underset{F'}{\mathrm{Max}} \iint R(f,d)\ d\,F'(f)\ d\,G'(g)\ ,$$

where F' and G' are distribution functions over Δ and Γ respectively. These distribution functions are complicated in form and will not be rigorously defined at present.

3. Two Modifications

Although both players distribute their forces over the n routes simultaneously the problem is simplified considerably by assuming the process is sequential in nature and that the game is a multi-stage game. The play is assumed to begin with a first stage at which each player selects a route. For simplicity each is assumed to select the same route, which is arbitrarily given the index one. The players then allocate a certain quantity of their resources to the route, Player A choosing f_1 and Player B choosing g_1 where $0 \leq f_1 \leq A$ and $0 \leq g_1 \leq B$. There are two consequences of this first choice. Player A receives a payoff of $R(f_1,g_1)$ and Player B receives $-R(f_1,g_1)$. At the same time, there is a reduction in the available resources of both players, with A being transformed into $T(f_1,g_1)$ and B becoming $S(f_1,g_1)$. The game continues in like fashion for (n-1) additional stages.

At the end of the n moves the total return to Player A of the game is

$$R = R(f,d)$$

$$= R(f_1, f_2, \ldots, f_n; g_1, g_2, \ldots, g_n)$$

$$= \sum_{i=1}^{n} R(f_i,g_i) \quad .$$

It is assumed that $R(f,g)$, $T(f,g)$ and $S(f,g)$ are continuous functions of f and g for all finite f and g and that these are functions only of the allocations and resource constraints, not of the route in question.

To further simplify the computational procedures we can represent the allocations of both players to any infiltration route in terms of the proportion of their total resources which each commits. Define the linear transformations

$$X = X(\Delta) = \frac{1}{A}(\Delta)$$

$$X = Y(\Gamma) = \frac{1}{B}(\Gamma).$$

Then the new sets X and Y are seen to be

$$X = \{x| \sum_{i=1}^{n} x_i \leq 1;\ x_i \geq 0\}$$

$$Y = \{y| \sum_{i=1}^{n} y_i \leq 1;\ y_i \geq 0\}.$$

and if U_n denotes the n-dimensional unit space, then $X \leq U_n$ and $Y \leq U_n$. The game can now be formulated as a process of allocating portions of the total force available to the various routes.

4. Modified Formulation

Using the simplifications introduced above the allocation game can now be reformulated in a convenient manner. Define the class of n-dimensional probability distribution functions D_n as

$$D_n(w) = D_n(w_1, w_2, \ldots, w_n)$$

$$= \text{Prob}(W_1 \leq w_1, W_2 \leq w_2, \ldots, W_n \leq w_n)$$

where D_n satisfies the following conditions:

(i) for any $w \varepsilon U_n$, $D_n(w)$ is a non-negative, real number

(ii) $D_n(0, 0, \ldots, 0) = 0$

$D_n(1, 1, \ldots, 1) = 1$

(iii) D_n is a non-decreasing function in w_i for $i = 1,2,\ldots,n$

(iv) If $\overline{1}$ is the n-dimensional sum vector, then for all real $\delta > 0$,

$$\lim_{\delta \to 0} D_n(W+\delta\overline{1}) = D_n(w).$$

When the return function is developed it will be shown that the value of the game, v_n, depends only upon two attrition coefficients and upon the initial resources of the two players, A and B. The sequence of functions

$$h_K(A,B) = v_k \quad k = 1, 2, \ldots, n,$$

can be used to reduce the problem of solving the original n-stage game to one of solving n single-stage continuous games. This reduction is made possible by a decomposition technique which is based on a fundamental principle of dynamic programming stating that an optimal policy for a sequential decision process has the property that whatever the initial state and initial decision of the process, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision. This principle of optimality has been shown by Bellman to be as valid for game purposes as it is for one-person decision processes [2]. Applying this principle to the present situation results in the following recurrence relations:

$$h_1(A,B) = \underset{F}{\mathrm{Max}}\ \underset{G}{\mathrm{Min}} \int_0^1 \int_0^1 R(x,y)\, dF(x)\, dG(y)$$

$$= \underset{G}{\mathrm{Min}}\ \underset{F}{\mathrm{Max}} \int_0^1 \int_0^1 R(x,y)\, dF(x)\, dG(y)$$

$$h_{k+1}(A,B) = \underset{F}{\mathrm{Max}}\ \underset{G}{\mathrm{Min}} \int_0^1 \int_0^1 [R(x,y) + h_k(T,S)] dF(x) dG(y)$$

$$= \underset{G}{\mathrm{Min}}\ \underset{F}{\mathrm{Max}} \int_0^1 \int_0^1 [R(x,y) + h_k(T,S)] dF(X) dG(y)$$

where $F, G \varepsilon D_n$.

The decomposition concept introduced above can be used to solve the general problem if the return function R(x,y) and the transformation functions T(x,y) and S(x,y) are well behaved and if certain other mathematical conditions are satisfied. These properties and conditions will be demonstrated after the form of the return function has been determined. Before attempting to derive an appropriate payoff some of the assumptions inherent in the formulation will be examined.

D. ASSUMPTIONS

There were a number of assumptions made, both tacitly and explicitly, in developing the game formulation of the problem. The major assumptions dealt with the tactical nature of the conflict along the infiltration routes and were based primarily on actual military experience.

The area of operations for both antagonists was assumed to be limited to the designated area of tactical responsibility of the counter-insurgent player. The infiltration network through this area was assumed to be finite and undirected, with an infinite capacity on all the arcs. The number of routes used for planning purposes by both participants was assumed to be the number of arcs in the minimum cut set of the network. Along these arcs, the combat activities of the insurgent player were limited to defensive operations, and the counter-insurgent player was limited specifically to the conduct of ambush operations.

The planning period for the problem was assumed to be short, not longer than one day. This assumption permitted the analysis of the long-range allocation problem as a sequence of independent, short-range problems, each constituting a separate play of the game. It was further assumed that any replacements or casualties affected only the resources available for allocation at the start of each play, and those resources were assumed to be limited. Since the planning period for most tactical operations of this type is usually short and since a military commander is limited in the forces at his disposal, these assumptions do not seem unreasonable.

The flow through the network was assumed to be measurable in terms of the number of men infiltrated during a planning period. Most infiltration operations in support of guerrilla operations rely almost exclusively on human bearers for the transport of supplies, especially if the terrain in the area is difficult or if the infiltration operations are being interdicted by the opposing side. This assumption permits an intuitively reasonable evaluation of the return to either side in a play of the game, which will be developed in the next section together with the transformation functions which affect the resources available to the players at each stage of the game.

III. THE RETURN FUNCTION

The formulation up to this point has utilized only a general specification of the payoff or return function with many of the properties of this function being assumed. The return function used in this study will now be developed and its properties explored.

Obtaining a realistic measure of effectiveness in conducting a counter-guerrilla campaign has consistently been a problem in itself. The traditional measures used to judge success in ground combat seldom apply to insurgent conflifts, since a successful counter-guerrilla campaign usually includes sociological and psychological programs which are not as important in conventional warfare. A true measure of effectiveness for counter-insurgency would probably take into account all of these factors.

The most common measure of effectiveness applied to the outcome of the purely military conflicts of recent counter-guerrilla operations has been the casualty ratio. A military commander today is presumed to be justified in sustaining heavy casualties to his own force if proportionately larger casualties are inflicted on the enemy, while a commander who suffers losses without inflicting greater harm on the adversary is judged a poor commander, regardless of the relative importance of the engagements in the overall conflict. Since the purpose of this study is to provide the military commander with a realistic planning model, the casualty ratio will be accepted as a measure of effectiveness without further justification. The technique used to determine a functional representation of the casualty ratio will be an application of Lanchester's equations.

A. LANCHESTER CONSIDERATIONS

The casualty rate of both forces involved in an ambush was modeled by Deitchman [5] using Lanchester's equations. Let the initial strength of the ambushing force be f and that of the ambushed force g. The differential equations relating the opposing forces are

$$\frac{df}{dt} = - \alpha fg$$

$$\frac{dg}{dt} = - \beta f$$

where α and β are the Lanchester attrition-rate coefficients for the forces of Player A and Player B, respectively, operating along some arbitrary infiltration route in the area of operations.

By multiplying both equations by appropriate factors, the left hand sides of the relations can be equated. Subsequent integration of both sides yields the relation

$$(f_o - f_t) = \frac{\alpha}{2\beta}(g_o^2 - g_t^2) ,$$

where f_t and g_t represent the strengths of each force at time t (with $f_o = f_i$ and $g_o = g_i$), and the condition for parity in a fight to the finish is given by

$$f_o = \frac{\alpha}{2\beta} g_o^{\;2} \;.$$

If in a conflict either initial force is less than that required for parity, that force will be assumed to lose the engagement.

In his study of ambush situations in guerrilla warfare Deitchman notes that the attrition rate coefficients usually differ by orders of magnitude. Both are basically kill probabilities, with

$$\beta = r_f p$$

where r_f is the rate of fire of the ambusher's weapons and p is the single-shot kill probability of each weapon in direct fire, and

$$\alpha = r_g \frac{A_{eg}}{A_f}$$

where r_g is the rate of fire of the ambushed forces weapons, A_{eg} is the single-round area of effectiveness of those weapons, and A_f is the area into which the force is firing. If rates of fire are equal on both sides $\alpha/2\beta$ can be on the order of 1/1000 or smaller [5].

B. CASUALTY FUNCTIONS

The preceding equations can now be used to determine the casualties for both sides involved in an ambush as a function of the initial force strengths. The combat is begun with Player B sending some of his force, g_o, along the ambushed infiltration route, and with Player A ambushing with some of his force, f_o. The combat is assumed to continue until the number of survivors of the ambushing force diminishes to some critical value m, or until the number of survivors of the ambushed force diminishes to n. The side whose strength first reaches the critical value loses; the other side wins the engagement. If τ is the time at which the losing force reaches this critical value, the casualties for each side can be easily determined. Define the casualties for the insurgent force as

$$C_B\ (f_o,g_o) = g_o - g_\tau$$

and those for the counter-insurgent force as

$$C_A\ (f_o,g_o) = f_o - f_\tau\ .$$

The expressions for f_τ and g_τ, the terminal force strengths for each side, can be determined directly from the relation derived in the preceding section.

From the definition of the critical values, m and n, it can be seen that these are the maximum force strengths at which either side will break contact and retire from the engagement. Since the use of positive critical values complicates the development considerably, it will be assumed that both m and n are zero. Then there are four different conditions of initial forces f_o and g_o which are of interest.

1. If either force meets no opposition, that is, if

$$f_o = 0 \quad \text{or} \quad g_o = 0$$

then there will be no engagment and no casualties and

$$C_A(f_o, g_o) = C_B(f_o, g_o) = 0 \ .$$

2. If the initial force ratio of the combatants is

$$f_o > \frac{\alpha}{2\beta} g_o^2$$

then the combat will result in a win for the ambushing force. From the definition of a win given earlier, this implies that the ambushed force is completely wiped out, that is, $g_\tau = 0$. The casualties are then

$$C_A(f_o, g_o) = \frac{\alpha}{2\beta} (g_o^2 - 0) = \frac{\alpha g_o^2}{2\beta}$$

$$C_B(f_o, g_o) = [g_o - 0] = g_o \ .$$

3. If the parity condition

$$f_o = \frac{\alpha}{2\beta} g_o^2$$

holds, then neither side will win. The two forces will exactly cancel each other, with the casualties suffered by each side being equal to the force employed, and

$$C_A(f_o,g_o) = f_o$$

$$C_B(f_o,g_o) = g_o \ .$$

4. If the ambushing force is inferior to the force being ambushed so that

$$f_o < \frac{\alpha}{2\beta} g_0^2$$

then the engagement will result in a loss for the ambushing force, and therefore

$$C_A(f_o,g_o) = f_o \ .$$

The casualties for the ambushed force can be determined by solving the equation

$$f_o = \frac{\alpha}{2\beta} (g_o^2 - g_\tau^2)$$

$$= \frac{\alpha}{2\beta} (g_o+g_\tau)(g_o-g_\tau)$$

$$= \frac{\alpha}{2\beta} \left[2g_o \ C_B(F_o,g_o) - [C_B(f_o,g_o)]^2\right]$$

The solution of this equation is

$$C_B(f_o,g_o) = g_o - \sqrt{g_0^2 - \frac{2\beta}{\alpha} f_o} \ .$$

When the allocations of the two players are made in terms of a proportion of their total resources committed to each route, the expected casualties resulting from such a commitment can also be determined as the fraction of the total force lost in the engagement. These proportions x_i and y_i of their total resources allocated by the players to the i^{th} infiltration route are

$$x_i = (1/A)\, f_o$$

$$y_i = (1/B)\, g_o \;,$$

and the condition of parity is now

$$x_i = \frac{\alpha}{2\beta} \frac{B^2}{A} y_i^2 \;.$$

It is obvious that the relation between x_i and y_i differs from the relations developed for f_o and g_o only by a constant of proportionality which depends on the total initial resources, A and B, of the two players.

The Lanchester attrition rate coefficients of the two forces, α and β, will be assumed constant over time. This assumption is not made purely out of mathematical necessity, since military forces tend to react during military encounters in ways which are characteristic of the units involved. The area occupied by an ambusher, the rate of fire of weapons, and the single-shot kill probabilities of those weapons are usually similar from engagement to engagement. Since the attrition coefficients are dependent on these factors the coefficients also tend to be constant over time.

When the attrition coefficients are constant, the factor of proportionality is a function of only the initial resource parameters, A and B and this factor can be defined as

$$\kappa = \kappa(A,B) = \frac{\alpha}{2\beta} \frac{B^2}{A} .$$

The casualty functions for Player A and B can now be determined for an ambush to which the players commit only portions of their total force. If the portion committed by Player A is x and that of Player B is y, the casualty functions for each side, $C_A(x,y)$ and $C_B(x,y)$, are summarized in Table I.

TABLE I

CASUALTY FUNCTIONS FOR $x,y \varepsilon [0,1]$

Range	$C_A(x,y)$	$C_B(x,y)$
x=0 or y=0	0	0
$x < \kappa y^2$	x	$y-\sqrt{y^2-(1/\kappa)\ x}$
$x = \kappa y^2$	x	y
$x > \kappa y^2$	κy^2	y

C. THE RETURN FUNCTION

Using the casualty functions developed above, an expression for the casualty ratio associated with the choices of the two players for the i^{th} infiltration route is given by

$$R(x_i, y_i) = \frac{C_B(x_i, y_i)}{C_A(x_i, y_i)}$$

for all $x, y > 0$.

If the minimax decision rule is applicable to this problem as was assumed earlier it would seem logical for Player A to wish to maximize this ratio, while Player B should wish to minimize it. A difficulty arises from the zero sum assumption made in the game formulation. The return function proposed above is a non-zero sum function, as the (negative) return to each side, the casualties suffered, is not provided by the opposing side. Therefore the total resources available at the end of a play of the game are less than those before the play.

One method of analyzing non-zero sum games is to introduce a dummy player who receives a payoff equivalent to the total resources lost by the other players [7]. This approach would complicate the current problem considerably, since a unique definition of the solution of an n-person game does not exist for n greater than two [3, 15]. Instead, it will be shown that the return function proposed, though not zero sum, can be adjusted to make the minimax principle apply. This can be accomplished by applying an extension of the fundamental theorem of continuous games.

1. <u>The Extended Theorem of Continuous Games</u>

The following theorem is an extension of a similar theorem for discrete (matrix) games.

Theorem. Let F and G be functions such that $F,G \varepsilon D_1$, where D_1 is the class of univariate probability distribution functions defined in Section II. If $M(x,y)$ and $N(x,y)$ are two arbitrary continuous functions over the closed unit square and if

$$\int_0^1 \int_0^1 N(x,y)\, dF(x)\, dG(y) \geq d > 0$$

for all $F,G \varepsilon D_1$, then

$$\operatorname*{Max}_{F} \operatorname*{Min}_{G} \frac{\int_0^1 \int_0^1 M(x,y)\, dF(x)\, dG(y)}{\int_0^1 \int_0^1 N(x,y)\, dF(x)\, dG(y)} = \operatorname*{Min}_{G} \operatorname*{Max}_{F} \frac{\int_0^1 \int_0^1 M(x,y)\, dF(x)\, dG(y)}{\int_0^1 \int_0^1 N(x,y)\, dF(x)\, dG(y)}.$$

The proof of the theorem involves a variation of a multiple approximation technique developed by Bellman [2]. The proof is lengthy and contributes little to the present development and consequently will not be detailed at the present time. It is included as Appendix A of this report.

2. Modified Casualty Function

The expected casualty functions can easily be modified to conform to the conditions of the theorem. It can be readily verified that both $C_A(x,y)$ and $C_B(x,y)$ are continuous functions over their range of definition and thus satisfy the first condition of the theorem. However, the second condition states that the integral of the function in the denominator must be positive over all possible values in its range. This condition is not satisfied if either

Player A or Player B uses as a strategy a probability step function with a single step at zero. Thus, for all $F, G \varepsilon D_1$

$$\int_0^1 \int_0^1 C_A(x,y)\,dI_o(x)\,dG(y) = \int_0^1 \int_0^1 C_A(x,y)\,dF(x)\,dI_o(y)$$

$$= 0,$$

where $I_o(x)$ is a discontinuous distribution function such that

$$I_o(0) = 0$$

$$I_o(x) = 1 \quad \text{for } x > 0.$$

However, the original casualty function $C_A(x,y)$ can be used to define a new function

$$C_A^*(x,y) = C_A(x,y) + d ,$$

where d is a positive real number. It can be readily seen that this new function satisfies the requirements for the denominator of the return ratio specified by the theorem. Although this new function was arbitrarily defined to make the problem tractable, it is not unreasonable to assume that the ambushing force experiences some small constant loss regardless of the level of hostilities. This loss is often a consequence of the area of operations itself and may be a result of disease or accidental causes.

3. Final Form

The return function which will be used in the final formulation and solution of the problem will be

$$R(x_i,y_i) = \frac{C_B(x_i,y_i)}{C_A^*(x_i,y_i)} .$$

This function will be applied recursively to the successive stages or routes of the planning problem in the manner outlined in Section II.

D. THE TRANSFORMATION FUNCTIONS

At each stage of the allocation game there is an alteration in the resources available to each player which is a function of the allocation made at that stage. This alteration has been represented symbolically with the transformation functions T(A,B) and S(A,B). The form of these transformation functions will now be determined.

When each player makes his choice at a stage of the game there is an upper limit to the forces he can commit at that stage which is the amount of uncommitted resources he still has. Let the uncommitted resources available to Player A at the start of stage i be a_i, and let those available to Player B at that stage be b_i. The reduction in these resources as a result of the allocation made by each player to stage i then determines the resources available for subsequent stages of the game. This transformation is then

$$\begin{aligned} a_{i-1} &= T(a_i,x_i) \\ &= a_i - x_i \end{aligned}$$

$$b_{i-1} = S(b_i, y_i)$$

$$= b_i - y_i$$

for $i = 1, 2, \ldots, n$ and

$$a_n = 1$$

$$b_n = 1.$$

All of the information necessary for the complete formulation is now available. In the next section the problem will be restated using the return functions and stage transformations outlined above, and the resulting equations will be used to determine the optimal allocation policy.

IV. THE METHOD OF SOLUTION

The individual returns and stage transformations developed in the preceding section will now be used to state the allocation problem in its final form. After showing that the method of decomposition can be applied to the problem, the n-stage allocation game will be solved and the optimal allocation strategies for each player will be determined. The general method of solution is to apply the decomposition principle outlined in Section II to transform the serial n-stage allocation problem into n equivalent single-stage allocation problems.

A. FINAL PROBLEM STATEMENT

The allocation problem discussed in the earlier sections can be briefly restated as follows. Two opponents, an insurgent and a counter-insurgent, operate in an area through which there are exactly n supply routes. The insurgent uses these supply routes to infiltrate men through the area, and the counter-insurgent uses the routes to ambush the insurgents. Both players wish to allocate their forces to the n routes in an optimal manner.

In an earlier section the problem was shown to be similar to a multi-stage game. Each stage i in the game is a move at which the two players simultaneously allocate a portion of their forces to operate on the i^{th} infiltration route. The allocations the two players make reduce the resources they have available for subsequent allocations and determine the returns they each receive from that route. This process continues for the n moves or stages which constitute a play of the game, and at the end of the play the players receive a return which is a function of the return from each stage of the game. Each player wishes to allocate his forces at each stage so as to maximize his overall return for a play of the game.

This sequential decision problem can be pictorially represented as in Figure 2. At each stage i of the game the input state variables are the current uncommitted resources, a_i and b_i, available to the counter-insurgent and the insurgent, respectively. The decision variables, x_i and y_i, are the allocations of each of the players, and the returns, C^*_{Ai} and C_{Bi}, are the casualties sustained by each. The stage transformations, T_i and S_i, are the reductions in available resources resulting from the allocations of each player.

By the principle of optimality, any optimal set of allocations for either player has the property that, whatever the decision made at any stage of the process, the remaining set of decisions must be optimal with respect to the outcome of that decision. This principle will be used to decompose the n-stage allocation game into n equivalent single stage games, and the solution to the overall game will be determined by recursively solving each of the single stage games.

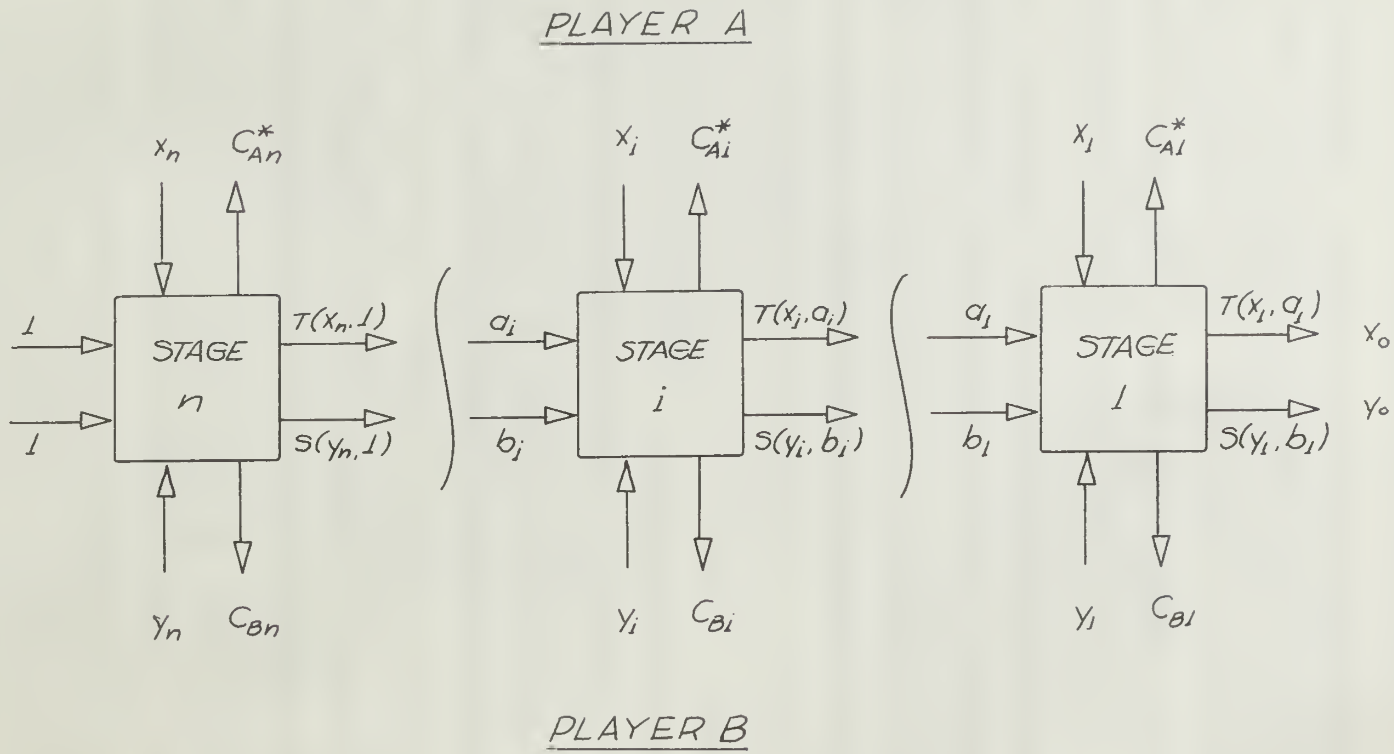

Figure 2. Flow Diagram of n-Stage Allocation Process

The recursive equations which will be used to solve the allocation game are similar to those of Section II, but these incorporate the return functions and stage transformations obtained in Section III.

The casualty function derived earlier for the insurgent force can be used to define a recurrence relation. Let the return to the insurgent at any stage be the sum of the individual return from the allocations made at that stage and the return to the insurgent at the preceding stage. Therefore,

$$C_{B1}(A,B) = C_B(x_1,y_1;A,B)$$

$$C_{Bi}(A,B) = C_B(x_i,y_i;A,B) + C_{Bi-1}[S(x_i,y_i;A,B)]$$

$$\text{for } i = 2,3,\ldots,n\ .$$

In a similar manner a sequence can be defined for the counter-insurgent as

$$C_{A1}(A,B) = C_A(x_1,y_1;A,B)$$

$$C_{A1}(A,B) = C_A(x_i,y_i;A,B) + C_{Ai-1}[T(x_i,y_i;A,B)]$$

$$\text{for } i = 2,3,\ldots,n$$

and

$$C^*_{Ai}(A,B) = C_{Ai}(A,B) + d \quad \text{for} \quad i = 1,2,\ldots,n\ .$$

The recurrence equations for the n-stage game can then be given by

$$h_1(A,B) = \underset{F}{\text{Max}}\ \underset{G}{\text{Min}}\ R_1(A,B)$$

$$= \underset{F}{\text{Max}}\ \underset{G}{\text{Min}}\ \frac{\int_0^1 \int_0^1 C_{B1}(A,B)dF(x_1)dG(y_1)}{\int_0^1 \int_0^1 C^*_{A1}(A,B)dF(x_1)dG(y_1)}$$

$$h_i(A,B) = \underset{F}{\text{Max}}\ \underset{G}{\text{Min}}\ R_i(A,B)$$

$$= \text{Max Min}\ \frac{\int_0^1 \int_0^1 C_{Bi}(A,B)dF(x_i)dG(y_i)}{\int_0^1 \int_0^1 C^*_{Ai}(A,B)dF(x_i)dG(y_i)}$$

where $C_{Bi}(A,B)$ and $C^*_{Ai}(A,B)$ are as defined above and where

$$T(x_i,y_i;A,B) = a_i - x_i$$

$$S(x_i,y_i;A,B) = b_i - y_i \qquad \text{for } i = 2,3,\ldots,n$$

and where

$$a_n = 1$$

$$b_n = 1$$

and

$$x_i,y_i,a_i,b_i \geq 0.$$

This set is similar to the usual recursive equations of dynamic programming, and its recursive solution, starting with $i = 1$ and continuing through $i = n$, yields the optimal n-stage return

$$v_n = h_n(A,B) = \underset{F}{\mathrm{Max}}\ \underset{G}{\mathrm{Min}}\ R_n(A,B)$$

where $R_n(A,B)$ is by definition the overall casualty ratio for the n routes. In addition, the solution yields the optimal strategies F_o and G_o for the two antagonists.

When the decomposition technique was first proposed in Section II it was assumed that its application was valid for the allocation problem being studied. Before attempting a solution using the decomposition technique its applicability will be verified by demonstrating the sufficient condition for decomposition given by Mitten [9].

B. GENERALIZED COMPOSITION

To justify the decomposition of the n-stage allocation process the crucial step of moving the optimization with respect to $x_{n-1},\ldots,x_1,y_{n-1},\ldots,y_1$ inside the n^{th} stage return must be achieved. A sufficient condition for accomplishing this change in the position of optimization has been given by Mitten [9]. Since the Mitten condition applies to pure maximization problems the condition will be demonstrated separately for each player.

The problem of the maximizing player, Player A, will first be considered for a two-stage process. The objective function at stage two is

$$h_2(A,B) = \underset{F}{\text{Max}}\ \underset{G}{\text{Min}} \frac{\int_0^1 \int_0^1 C_{B2}(A,B)dF(x_2)dG(y_2)}{\int_0^1 \int_0^1 C^*_{A2}(A,B)dF(x_2)dG(y_2)}$$

$$= \underset{F}{\text{Max}}\ \underset{G}{\text{Min}} = \frac{\int_0^1 \int_0^1 [C_B(x_2,y_2,a_2,b_2)+C_B(x_1,y_1,a_1,b_2-y_2)]dF(x_2)dG(y_2)}{\int_0^1 \int_0^1 [C_A(x_2,y_2,a_2,b_2)+C_A(x_1,y_1,a_2-x_2,b_1)+d]dF(x_2)dG(y_2)}$$

Let G_o be the distribution function which optimizes this function for Player B. Then

$$h_2(A,B) = \underset{F}{\text{Max}} \frac{\int_0^1 \int_0^1 [C_B(x_2,y_2,a_2,b_2)+C_B(x_1,y_1,a_1,b_2-y_2)]dF(x_2)dG_o(y_2)}{\int_0^1 \int_0^1 [C_A(x_2,y_2,a_2,b_2)+C_A(x_1,y_1,a_2-x_2,b_1)+d]dF(x_2)dG_o(y_2)}$$

$$= \underset{F}{\text{Max}}\ H_2(A,B,G_o).$$

But since $C_A(x_1,y_1)$ is monotonic nonincreasing in x_1, and $C_B(x_1,y_1)$ is monotonic nondecreasing in x_1, $H_2(A,B,G_o)$ is monotonic nondecreasing in x_1, which is the sufficient condition derived by Mitten for optimal decomposition of a maximization process. This result can be easily extended to the n-stage case, and the validity of decomposition for Player A is verified.

Consider the problem now of the minimizing player, Player B. Recall from the theorem of Section III

$$h_2(A,B) = \underset{F}{\text{Max}}\ \underset{G}{\text{Min}}\ R_2(A,B)$$

$$= \underset{G}{\text{Min}}\ \underset{F}{\text{Max}}\ R_2(A,B)$$

Let F_o be the distribution which maximizes $R_2(A,B)$ for Player A. Then

$$h_2(A,B) = \underset{G}{\text{Min}}\ H_2(A,B,F_o)$$

$$= -\left[\underset{G}{\text{Max}}\ [-H_2(A,B,F_o)]\right] .$$

But, considering the form of $H_2(A,B,F_o)$ given in the derivation for the maximizing player it can be seen that, since $C_B(x_1,y_1)$ is monotonically nonincreasing in y_1, and since $C_A(x_1,y_1)$ is monotonically nondecreasing in y_1, $H_2(A,B,F_o)$ is monotonically nonincreasing in y_1. Therefore, $-H_2(A,B,F_o)$ is monotonically nondecreasing in y_1, which is again a sufficient condition for decomposition. The validity of the decomposition technique is therefore verified for the n-stage allocation game being studied.

C. SOLUTION

The n-stage allocation game will now be solved by solving the recursive equations given earlier. For i = 1,

$$h_1(A,B) = \underset{F}{\text{Max}}\ \underset{G}{\text{Min}}\ R_1(A,B)$$

$$= \underset{F}{\text{Max}}\ \underset{G}{\text{Min}} \frac{\int_0^1 \int_0^1 C_B(A,B)\,dF(X_1)\,dG(y_1)}{\int_0^1 \int_0^1 C_A^*(A,B)\,dF(x_1)\,dG(y_1)}$$

$$= \text{Max Min} \frac{\int_0^1 \int_0^1 C_B(x_1,y_1;\kappa)\,dF(x_1)\,dG(y_1)}{\int_0^1 \int_0^1 [C_A(x_1,y_1;\kappa)+d]\,dF(x_1)\,dG(y_1)}$$

where

$$\kappa = \kappa(A,B) = \frac{\alpha}{2\beta}\frac{B^2}{A} .$$

Also, as was shown earlier,

$$C_B(x,y;\kappa) = \begin{cases} 0 & , y = 0 \\ y & , y \leq \sqrt{\frac{1}{\kappa}}\, x \\ y-\sqrt{y^2-(1/\kappa)x} & , y > \sqrt{\frac{1}{\kappa}}\, x . \end{cases}$$

It is readily obvious that there exists a unique minimum for $R_1(A,B)$ which is attained at $y_1 = 0$. The distribution function G_1^o, where G_1 is the marginal distribution of x_1 in the distribution function G_1 is given by

$$G_1^o = I_o(y_1) = \begin{cases} 0 & \text{for} \quad y_1 = 0 \\ 1 & \quad\quad y_1 > 0 . \end{cases}$$

Thus

$$h_1(A,B) = \underset{F}{\text{Max}} \frac{\int_0^1 \int_0^1 C_B(x_1,y_1;\kappa)\ dF(x_1)dI_o(y_1)}{\int_0^1 \int_0^1 [C_A(x_1,y_1;\kappa)+d]\,dF(x_1)dI_o(y_1)}$$

$$= \underset{F}{\text{Max}}\ \frac{0}{d}$$

$$= 0\ .$$

The optimal strategy for Player B at the first stage is to employ no forces, with probability one. This implies that the insurgent should not use the first infiltration route at all if he wishes to minimize the casualty ratio over all of the routes. And, due to the construction of the return function, the optimal strategy for Player A at stage one is any possible strategy. Due to the unique minimum achieved by Player B, Player A cannot affect the outcome of the first stage.

It becomes readily apparent that this result will hold for the entire problem. Consider

$$h_i(A,B) = \underset{F}{\text{Max}}\ \underset{G}{\text{Min}} \frac{\int_0^1 \int_0^1 C_{Bi}(A,B)dF(x_i)dG(y_i)}{\int_0^1 \int_0^1 C^*_{Ai}(A,B)dF(x_i)dG(y_i)}$$

$$= \underset{F}{\text{Max}}\ \underset{G}{\text{Min}} \frac{\int_0^1 \int_0^1 [C_B(x_i,y_i,\kappa)+C_{Bi-1}(A,B)]\,dF(x_i)dG(y_i)}{\int_0^1 \int_0^1 [C_A(x_i,y_i,\kappa)+C_{Ai-1}(A,B)+d]\,dF(x_i)dG(y_i)}$$

Since Player B can always limit the casualties he has sustained on all of the previous routes to zero, the i^{th} stage relation becomes

$$h_i(A,B) = \underset{F}{\text{Max}}\ \underset{G}{\text{Min}} \frac{\int_0^1 \int_0^1 C_B(x_i,y_i;\kappa)dF(x_i)dG(y_i)}{\int_0^1 \int_0^1 [C_A(x_i,y_i;\kappa) + d]dF(x_i)dG(y_i)}$$

which is the problem which was solved for the first stage.

The solution to the game is therefore easily seen. If the objective of the insurgent is to minimize the casualty ratio in favor of the counter-insurgent, the optimal strategy is to send, with probability one, no forces through the infiltration network. The optimal strategy for the counter-insurgent is any strategy he desires. The value of the game, the expected casualty ratio in favor of the counter-insurgent, is zero. The implications which can be made about the model from this result will be investigated in the succeeding chapter, and some modifications will be proposed.

V. THE MODIFIED PROBLEM AND SOLUTION

In the preceding section it was shown that the optimal strategy for the insurgent in the allocation game was to avoid the use of the network entirely if his objective was to minimize the casualty ratio in favor of the ambush or counter-insurgent force. The expected value of the casualty ratio is then zero, which is the value of the game. This result, though mathematically correct for the model given, is not appealing since the insurgent may have goals which are not adequately represented by the payoff function which was used. In this section an additional modification will be made in the return function to represent the case in which the insurgent wishes to maximize the men infiltrated through the area. A solution for the new game will be determined, and the solution will be compared with the previous result.

A. MODIFICATION OF INSURGENT RETURN

The result which was obtained in the preceding section is the obvious solution for an insurgent with complete freedom of choice in the actions he can take. One of the major tenets of guerrilla warfare is for the insurgent to maintain the initiative and to avoid contact when this initiative is lost. Thus an insurgent with a set of infiltration routes which are being ambushed or patrolled by the opposing force would avoid using those routes if he were free to do so.

The insurgent, however, may not always be able to completely abandon such an infiltration network. As was noted in the introduction of this paper, the level of hostilities which the insurgent can maintain in his chosen area of operations may depend to a large extent on the amount of men and materiel which can be infiltrated from the insurgent base areas or sanctuaries. In such cases the insurgent may continue to use routes which are being heavily interdicted by the opposing force. This has been the result in the Vietnam conflict in which the insurgent forces have continued and in some instances increased the use of their supply routes through the border areas of Laos and South Vietnam despite intense aerial interdiction [13]. It will be shown that the general game formulation used to analyze the original allocation problem can be extended to cover these new situations.

One method of extending the present model would be to place a lower bound on the forces which the insurgent could commit to the infiltration routes. This approach would be only partially successful, since it can be shown that the expected casualty function for Player B, $C_B(x,y)$, is convex in y for all allocations x of Player A. Thus the insurgent would profit by committing only those forces required by the constraint. Although the solution of such a game would provide the optimal allocation for those forces among the different infiltration routes, it would not determine what total force allocation would be optimal unless the game were solved for all possible lower bounds. Furthermore, such a model would not accurately reflect the true goal of the insurgent which is not to minimize casualties but to maximize the number of men successfully infiltrated.

A better method is to redefine the return function to accurately reflect the outcomes which the insurgent wishes to optimize. There are essentially two results from a play of the game. For each allocation made by Player B some fraction may traverse the network successfully. In addition, some portion is lost in casualties. The insurgent wishes to maximize the men through the system and minimize the casualties lost. Although these two objectives are usually incompatible there may be some tradeoff between them, a measure of which can be gained with the weighted difference of the two. Therefore, let the "worth" to the insurgent of one man who successfully travels the network be c_1 and let the "cost" of one casualty to the force be c_2. Then a candidate return function for the insurgent, the minimizing player, could be

$$C_B^*(x,y) = c_2 C_B(x,y) - c_1[y - C_B(x,y)],$$

where x and y are the allocations of the two players to the route in question and $C_B(x,y)$ is the casualty function defined earlier. If, for convenience, it is assumed that the cost c_2 of one casualty is half the worth c_1 of a successful infiltration, then the expression can be reduced to

$$C_B^*(x,y) = C_B(x,y) - y \ .$$

This new individual return $C_B^*(x,y)$ is always non-positive and Player B will seek to minimize it over his range of choice $y \in [0,1]$. Although this new return no longer represents a pure casualty function, it can be used in much the same manner as the original $C_B(x,y)$.

It is easily shown that the revised return function, $C_B^*(x,y)$, has the same properties with respect to Player B as the earlier casualty function, namely continuity. If the individual return for Player A, $C_A^*(x,y)$, is assumed to be the same expected casualty function defined earlier, then the two functions can be used to define a new criterion

$$R^*(x_i,y_i) = \frac{C_B^*(x_i,y_i)}{C_A^*(x_i,y_i)}$$

where, as before, x_i and y_i are the respective allocations of Player A and Player B to the i^{th} infiltration routes, with these allocations representing portions of the total available force. The revised individual return functions are tabulated in Table II, where x and y are arbitrary allocations.

The new multi-stage game is stated using recurrence equations similar to those of Section IV, but utilizing the new criterion function $R^*(x,y)$. Define

$$C^*_{A,1}(x,y) = C^*_A(x_1,y_1)$$

$$C^*_{A,i}(x,y) = C^*_A(x_i,y_i) + C^*_{A,i-1}(T(A,B))$$

$$i = 2,\ldots,n$$

$$C^*_{B,1}(x,y) = C^*_B(x_1,y_1)$$

$$C^*_{B,i}(x,y) = C^*_B(x_i,y_i) + C^*_{B,i-1}(S(A,B))$$

$$i = 2,\ldots,n\ .$$

where $T(A,B)$ and $S(A,B)$ are as defined earlier. Then the recurrence equations are

$$h^*_1 = \underset{F}{\text{Max}}\ \underset{G}{\text{Min}}\ R^*_1(x,y)$$

$$= \underset{F}{\text{Max}}\ \underset{G}{\text{Min}}\ \frac{\int_0^1\int_0^1 C^*_B(x_1,y_1)dF(x_1)dG(y_1)}{\int_0^1\int_0^1 C^*_A(x_1,y_1)dF(x_1)dG(y_1)}$$

$$h^*_i = \underset{F}{\text{Max}}\ \underset{G}{\text{Min}}\ R^*_i(x,y)$$

$$= \underset{F}{\text{Max}}\ \underset{G}{\text{Min}}\ \frac{\int_0^1\int_0^1 C^*_{B,i}(x,y)dF(x_i)dG(y_i)}{\int_0^1\int_0^1 C^*_{A,i}(x,y)dF(x_i)dG(y_i)}$$

$$\text{for } i = 2,3,\ldots,n\ .$$

The same mathematical arguments which were used to justify the decomposition of the original game can be used on the modified game to the same end. The recursive solution of this new game will yield the optimal strategies for both players and the value of the game.

TABLE II

REVISED INDIVIDUAL RETURN FUNCTIONS FOR $x,y \varepsilon [0,1]$

Range	$C_A^*(x,y)$	$C_B^*(x,y)$	$R^*(x,y)$
$x = 0$	d	$-y$	$-\frac{y}{d}$
$x < \kappa y^2$	$x + d$	$-\sqrt{y^2-(1/\kappa)x}$	$\frac{-\sqrt{y^2-(1/\kappa)x}}{x + d}$
$x = \kappa y^2$	$x + d$	0	0
$x > \kappa y^2$	$\kappa y^2 + d$	0	0
$y = 0$	d	0	0

where $\kappa = \kappa(A,B) = \frac{\alpha}{2\beta} \frac{B^2}{A}$

B. SOLUTION

In the solution which follows a great deal of use will be made of a class of discontinuous distribution functions first introduced in Section III. These functions will be rigorously defined, and some conditions on the factor of proportionality κ will be investigated before a solution is attempted.

1. Step Functions With One Step

Suppose there is a finite increasing sequence of points of [0,1]

$$x_1 < x_2 < \dots < x_n$$

such that a distribution function F has discontinuities at each of these n points but is constant elsewhere. That is

$$F(u) = F(v) \quad \text{if} \quad x_i < u,v < x_{i+1} .$$

Then F will be called a step function with n steps. The distribution which will be especially useful will be the step functions of one step. Such distributions will be written as I_c, which will denote a distribution function which has a step of one unit at the point c. Thus

$$\begin{aligned} I_c(x) &= 0 \qquad x < c \qquad c > 0 \\ &= 1 \qquad x \geq c \end{aligned}$$

and

$$I_o(0) = 0$$

$$I_o(x) = 1 \qquad x > 0 .$$

It can be easily shown that any step function of n steps F(x) can be written as a sum of n single step distribution function such that

$$F(x) = w_1 I_{x_1} + w_2 I_{x_2} + \dots + w_n I_{x_n}$$

where $x_1, x_2, \ldots, x_n$ are the step points of the original distribution and w_i is a constant such that

$$w_1 = F(x_1)$$

$$w_i = F(x_i) - F(x_{i-1}) \qquad i > 1.$$

2. Parametric Considerations

Two functions which will figure in the subsequent solution are the individual returns when both players employ their entire resources along a single infiltration route. These returns are determined by the initial resources of the two players, A and B, and the Lanchester attrition rate coefficients, α and β, which determine the factor of proportionality κ where

$$\kappa = \frac{\alpha}{2\beta} \; \frac{B^2}{A} .$$

The behavior of the individual returns $C_A^*(1,1)$ and $C_B^*(1,1)$, where each player commits his entire force, will be studied for three cases.

(1) $\kappa > 1$.

If this is the case, then an engagement to which both players commit their entire initial resources will always result in a win for Player B, the insurgent, and the functions of interest are

$$C_A^*(1,1) = 1 + d$$

$$C_B^*(1,1) = -\sqrt{1 - (1/\kappa)}$$

$$R^*(1,1) = -\frac{\sqrt{1 - (1/\kappa)}}{1 + d} .$$

(2) $\kappa < 1$

The result in this case would be a win for Player A, for the same reasons given above. Therefore

$$C_A^*(1,1) = \kappa + d$$

$$C_B^*(1,1) = 0$$

$$R^*(1,1) = 0 .$$

(3) $\kappa = 1$

This condition implies that an engagement to which both players commit their entire forces will result in a draw, that is, neither side will lose, and

$$C_A^*(1,1) = 1 + d$$

$$C_B^*(1,1) = 0$$

$$R^*(1,1) = 0 .$$

This final case will be assumed in the subsequent solution process which implies that the two forces are tactically equivalent at the start of the game in the sense of equivalence defined by Lanchester.

3. Specialized Solution

The problem at the first stage of the game is to solve

$$R_1^* = \underset{F}{\text{Max}}\ \underset{G}{\text{Min}}\ R_1^*(x,y)$$

$$= \underset{F}{\text{Max}}\ \underset{G}{\text{Min}}\ \frac{\int_0^1 \int_0^1 C_B^*(x_1,y_1)dF(x_1)dG(y_1)}{\int_0^1 \int_0^1 C_A^*(x_1,y_1)dF(x_1)dG(y_1)} .$$

Consider the problem first of Player B with regard to the numerator of the return function. The individual return, $C_B^*(x_1,y_1)$ is minimized by maximizing y_1 for all allocations x_1 of Player A. The maximum value of y_1 is one, and therefore

$$\underset{G}{\text{Min}} \int_0^1 \int_0^1 C_B^*(x_1,y_1)dF(x_1)dG(y_1) = \int_0^1 \int_0^1 C_B^*(x_1,y_1)dF(x_1)dI_1(y_1)$$

$$= \int_0^1 C_B^*(x_1,1)dF(x_1) .$$

Now consider the insurgent's problem with respect to the denominator, $C_A^*(x_1,y_1)$. Since the return to Player B is nonpositive for all allocations of the two players, and since he wishes to minimize the overall return function $R_1^*(x,y)$, he wishes to minimize the denominator $C_A^*(x_1,y_1)$, which is positive for all values in its range. But this minimum is accomplished by minimizing his allocation, y_1, regardless of the actions of his opponent. In particular,

$$\operatorname*{Min}_{G} \; C_A^*(x_1,y_1)dF(x_1)dG(y_1) = \int_0^1\int_0^1 C_A^*(x_1,y_1)dF(x_1)dI_o(y_1)$$

$$= \int_0^1 C_A^*(x_1,0)dF(x_1) \; .$$

Obviously, no pure distribution will accomplish the overall minimization of the return function $R_1^*(x,y)$ for Player B. It can be shown that the optimal solution under the assumption that $\kappa = 1$ is for the insurgent to play a mixed strategy, with regard to the first route, of

$$G^o(y_1) = (1-\phi_1)I_o(y_1) + \phi_1 I_1(y_1) \quad 0 \leq \phi_1 \leq 1 \; .$$

If this strategy is used, then

$$\operatorname*{Min}_{G} \frac{\int_0^1\int_0^1 C_B^*(x_1,y_1)dF(x_1)dG(y_1)}{\int_0^1\int_0^1 C_A^*(x_1,y_1)dF(x_1)dG(y_1)} = \frac{\int_0^1\int_0^1 C_B^*(x_1,y_1)dF(x_1)dG^o(y_1)}{\int_0^1\int_0^1 C_A^*(x_1,y_1)dF(x_1)dG^o(y_1)}$$

$$= \frac{\phi_1 \int_0^1 C_B^*(x_1,1)dF(x_1)}{(1-\phi_1)d + \phi_1 \int_0^1 C_A^*(x_1,1)dF(x_1)}$$

$$\equiv H(x_1) \; .$$

The counter-insurgent player then wishes to maximize $H(x_1)$ as defined above, by choosing some distribution function from D_1. The same method of analysis used for Player

F will be used to determine the optimal strategy for Player B. Since he wishes to maximize the overall function he will wish to maximize the numerator, $C_B^*(x_1,1)$. He can increase the casualties to the insurgent by increasing the ambushing force, x_1, along the infiltration route in question, and

$$\underset{F}{\text{Max}}\ \phi_1 \int_0^1 C_B^*(x_1,1)dF(x_1) = \phi_1\ \underset{F}{\text{Max}} \int_0^1 C_B^*(x_1,1)dF(x_1)$$

$$= \phi_1 \int_0^1 C_B^*(x_1,1)dI_1(x_1)$$

$$= \phi_1 C_B^*(1,1)\ .$$

where $I_1(x)$ is a step-function as defined earlier. At the same time, Player A wishes to minimize his own casualties, $C_A^*(x_1,1)$, which can be done by minimizing his allocation. Thus

$$\underset{F}{\text{Min}}\ \phi_1 \int_0^1 C_A^*(x_1,1)dF(x_1) = \phi_1 \int_0^1 C_A^*(x_1,1)dI_o(x_1)$$

$$= \phi_1 d\ .$$

As above, the optimal strategy $F^o(x_1)$ for Player A assuming $\kappa = 1$ is to play a combination of the two strategies $I_o(x_1)$ and $I_1(x_1)$, and

$$F^o(x_1) = (1-\psi_1)I_o(x_1) + \psi_1 I_1(x_1), \text{ for } 0 \leq \psi_1 \leq 1\ .$$

Based on this strategy

$$\operatorname*{Max}_{F} H(x_1)=\frac{\phi_1 \int_0^1 C_B^*(x_1,1)\,dF^O(x_1)}{(1-\phi_1)d+\phi_1 \int_0^1 C_A^*(x_1,1)\,dF^O(x_1)}$$

$$=\frac{\phi_1\psi_1 \; C_B^*(1,1) + \phi_1\psi_1 - \phi_1}{\phi_1\psi_1 \; C_A^*(1,1) + d - \phi_1\psi_1 d} .$$

But it was shown above that, under the assumption that $\kappa = 1$,

$$C_A^*(1,1) = 1 + d$$

$$C_B^*(1,1) = 0 .$$

Therefore, substituting above

$$h_1^* = \operatorname*{Max}_{F} H(x_1)$$

$$= \frac{\phi_1\psi_1 - \phi_1}{\phi_1\psi_1 + d} .$$

The second stage problem is then

$$h_2^* = \operatorname*{Max}_{F} \operatorname*{Min}_{G} \frac{\int_0^1\int_0^1 C_{B,2}^*(x,y)\,dF(x_2)\,dG(y_2)}{\int_0^1\int_0^1 C_{A,2}^*(x,y)\,dF(x_2)\,dG(y_2)}$$

$$= \operatorname*{Max}_{F} \operatorname*{Min}_{G} \frac{\int_0^1\int_0^1 \left[C_B^*(x_2,y_2)+\phi_1\psi_1-\phi_1\right] dF(x_2)\,dG(y_2)}{\int_0^1\int_0^1 \left[C_A^*(x_2,y_2)+\phi_1\psi_1+d\right] dF(x_2)\,dG(y_2)}$$

By the same [illegible] is used at stage [illegible], it can be shown that the optimal strategies for the two players are

$$F^{\circ}(x_2) = (1-\psi_2)\, I_0(x_2) + \psi_2 I_1(x_2)$$

$$G^{\circ}(y_2) = (1-\phi_2)\, I_0(y_2) + \phi_2 I_1(y_2) \; .$$

A similar result will occur for each stage of the game, and the optimal strategies will be for each player to play

$$F^{\circ}(x_i) = (1-\psi_i)\, I_0(x_i) + \psi_i I_1(x_i) \qquad 0 \le \psi_i \le 1$$

$$G^{\circ}(y_i) = (1-\phi_i)\, I_0(y_i) + \phi_i I_1(y_i) \qquad 0 \le \phi_i \le 1 \; .$$

where $F^{\circ}(x_i)$ is the marginal distribution for x_i in the joint distribution $F^{\circ}(\bar{x})$, and where $G^{\circ}(y_i)$ is the marginal distribution for y_i in $G^{\circ}(\bar{y})$.

The values of ψ_i and ϕ_i are determined in the following manner. Since the numbering system for the [illegible] is arbitrary and since all [illegible] are equally useful and accessible to each side, the condition of symmetry would [illegible]

$$\psi_1 = \psi_2 = \ldots \psi_n$$

and

$$\phi_1 = \phi_2 = \ldots \phi_n \; .$$

Furthermore, if $F^{\circ}(\bar{x})$ and $G^{\circ}(\bar{y})$ are bona fide distribution functions

$$\psi_1 + \psi_2 + \ldots + \psi_n = 1$$

and

$$\phi_1 + \phi_2 + \ldots + \phi_n = 1.$$

The solution to these sets of equations is

$$\psi_i = \phi_i = \frac{1}{n}\ .$$

Therefore the optimal strategy for <u>each</u> player in the case where $\kappa = 1$, is <u>to commit his entire force to a single infiltration route and to choose that route in a completely random fashion</u>.

The value for h_n^*, which is the value of the game, can be found from

$$h_n^* = \frac{\sum_1^n (\phi_i\psi_i - \phi_i)}{\sum_1^n (\phi_i\psi_i + d)}$$

$$= \frac{1 - n}{1 + n^2 d}\ .$$

The true meaning of this value is not clear since the modified objective function was of composite form. The expected casualty ratio can be calculated easily. The probability that both players choose the same route on any given play of the game is $\psi_i \times \phi_i = 1/n^2$ since each player chooses independently. In such an engagement both forces will be destroyed, and the ratio will thus be B/A. Therefore, the mean ratio of casualties is

[illegible]

[illegible]

If [illegible] values for δ and θ are [illegible] for $x = 1$,

[illegible]

and

[illegible]

Thus there is a small increase from the [illegible] ratio of [illegible] before [illegible]. At the same time, the [illegible] network for the [illegible] is greatly [illegible], [illegible] the [illegible] flow [illegible]

[illegible] (flow through network) = 1 − [illegible]

= 1 − [illegible]

where [illegible] the [illegible] system. This is a [illegible] the [illegible] of the preceding [illegible].

In the next section the importance of the assumptions made in the development will be examined, and the sensitivity of the solutions to changes in these assumptions will be investigated. In addition, some possible extensions of the general model, and some open problems will be proposed.

VI. CONCLUSION AND RECOMMENDATIONS

The original objective of this study as outlined in the introductory remarks was the development of a realistic allocation model to be used in conducting ambush interdiction of infiltration routes. A game theoretic model was then proposed, and a method for obtaining the optimal allocation strategies for the two players was presented. This method was then used to determine those best strategies for two special cases. In this section the degree to which the original objective was accomplished will be examined and the importance of some of the major assumptions will be discussed. Also, the manner in which the model can be extended to other situations will be mentioned, and some areas of future work will be suggested.

A. ASSUMPTIONS

The model which has been developed in the preceding sections does provide a method of allocating the forces of two opponents in an infiltration situation which optimizes the allocations of each under certain assumptions. The significance of the results, however, depends to a large degree on the nature of the assumptions made in deriving them. Two of the major assumptions which determined the form of the results were that a gaming model was appropriate for the situation being studied and that the ratio of the rational behavior of the players and a reasonable criterion or rationale to use in judging the outcomes.

The theory of games is founded on the properties of the minimax decision rule, properties first demonstrated by von Neumann and Morgenstern [17]. This rule is considered overly conservative and unreasonable by many, since it seeks to guard against the worst possible outcome of a game [8]. However, the allocation problems being studied are often characterized by such a pessimistic attitude on the part of the military or political leaders involved. In military conflict decisions involve the allocation and possible loss of human lives, and such decisions are often more conservative than decisions made regarding less sensitive resources. Therefore, the implicit acceptance of the minimax decision rule in deciding to develop a game theoretic allocation model is not unreasonable.

The most sensitive assumption was that the ratio of the engagement results was a realistic criterion for the game model. This form of payoff is not common in the analysis of non-zero-sum games [2, 7]. However, most solution concepts for such games assume at least the possibility of cooperation between the players of the game. The nature of the conflict between insurgent and counter-insurgent in the situation being modeled does not permit that cooperation.

This ratio or criterion function was first proposed by Isaacs [1] as a possible means of evaluating discrete non-zero sum games. The theorem stated in Section III and proved in Appendix A shows that a similar ratio can be used for continuous non-zero sum games, but whether or not to accept this criterion is a matter of individual taste. Its use was originally proposed because the classic criteria did not apply for the reason mentioned above and because the desired return function, the ratio of expected casualties, was itself a ratio. However, with the extension of the original method to account for the major goal of the insurgent, that of maximizing the men successfully infiltrated through the area of operations, the desirability of the ratio criterion is not easily demonstrated. Though the new criterion function does attempt to model the individual return to the insurgent it ignores the real goal of the counter-insurgent which is the interdiction of the infiltration operations.

Consider the two solutions which were determined in this paper for the two forms of the criterion function. The solution of the first case yielded a ratio of casualties of zero, with no infiltration being accomplished by the insurgent. For the second case the solution yielded a ratio of expected casualties which was considerably greater than unity with the assumption of tactical equivalence of the two forces, and consequently represented a significant improvement in the individual return to the counter-insurgent player. But at the same time, the number of men which the insurgent could expect to successfully traverse [illegible] representing a [illegible] of [illegible] non-[illegible]. [illegible], except the [illegible] to the counter-insurgent player was [illegible] to the [illegible] in the first, the first [illegible] represented complete [illegible] while the second [illegible] [illegible]. [illegible] function [illegible] for the counter-insurgent player in the [illegible] should be modified to more [illegible] to reflect the [illegible] goal of [illegible]. This modification could possibly be made in a manner similar to that used for the insurgent in the second [illegible].

The [illegible] advantage [illegible] the use of the criterion [illegible] is this flexibility [illegible] of the [illegible] returns for the two players. The [illegible] limitations on these return functions are that they be continuous and that the [illegible] function in [illegible]. [illegible], however, [illegible] the [illegible] becomes computationally difficult [illegible] return functions and the value of the game [illegible] in the [illegible] in [illegible] as the [illegible] returns for either player.

C. [illegible]

The [illegible] model can be used to analyze situations which differ considerably from the [illegible].

nature of the original allocation process. In addition, some infiltration situations which are not covered by the present model can be covered by the relaxation of some of the assumptions made in the original formulation.

1. Ambush Versus Meeting Engagements

In some situations the operations of the counter-insurgent may not be limited to the ambushing of infiltration routes, but may include offensive operations of different types. In these situations the counter-insurgent must allocate forces among a number of dissimilar alternatives, and the need for a planning model is increased. These non-homogeneous allocation processes can be analyzed with the present model by allocating all forces to a type of operation at a single stage of the game. The return for that stage would be the value of a sub-game of the type discussed in this paper, which would allocate forces between all feasible alternatives of a single type. Return functions for situations other than ambush could be determined using Lanchester models or other means.

2. The Critical Survival Level

One of the assumptions made in developing the expected casualty functions for the present model was that the critical survival levels m and n were both zero. These maximum survival levels at which a side will break contact are seldom zero in guerrilla conflicts, especially for the guerrilla force. Non-zero critical survival levels could be included in the present model by adjusting the stability [illegible]. One method of accomplishing this would be to [illegible] the effective [illegible] for each zone, that is, the [illegible] critical survival level [illegible] that zone, and to [illegible] the attrition coefficients accordingly.

Other refinements could be made in the [illegible] functions to [illegible] attrition coefficients, such as those [illegible] [15], and to include [illegible] rates for each [illegible]. The stochastic forms of Lanchester's equations could also be investigated and the attrition process analyzed as a stochastic [illegible] process.

2. Partial Information

The model as developed assumed that [illegible] players [illegible] of the opponent's allocation strategy. In combat, however, there is usually a continuous flow of battlefield information and intelligence to the military commanders involved. [illegible] opponent's [illegible] method of [illegible] to current tactical doctrine. The applicability of the allocation model developed could be increased by formulating the game as a game of partial information, with varying degrees of accuracy [illegible] for the information available to either or both players.

C. [illegible] SUMMARY

Even the present model is in a [illegible] satisfactory state of development. The most important [illegible] is the determination of the solution of the game in the preceding section when the opposing forces are not strategically equivalent, that is, when $\kappa \neq 1$. The optimal strategies in such a case are not of the simple form shown, and a detailed analysis of their form is needed. The variation of the optimal strategies for the players with changes in the initial forces for each could then be determined.

Additional problems have already been noted in the discussion of the individual return functions and in the extensions mentioned above. Furthermore, the general nature in which the game was formulated may permit the modeling of other, possibly non-military, allocation situations if the form of the criterion function is reasonable.

D. CONCLUSION

In this thesis, an allocation problem associated with counter-infiltration operations in guerrilla warfare has been investigated and a game theoretic planning model has been proposed. In addition, the model has been used to determine the optimal allocation policy for two specialized cases of the basic conflict situation. It is hoped that this study will be useful to planners involved in counter-infiltration and will generate interest in the further investigation of the problem.

PROOF. For simplicity we shall abbreviate $F \varepsilon D$ and $G \varepsilon D$ as F and G, and $dF(x)$ and $dG(x)$ as dF and dG. Define a function

$$L^*(x,y) = \gamma L(x,y)$$

where

$$\frac{1}{\gamma} = \max_F \max_G \int_0^1 \int_0^1 L(x,y)\,dF\,dG + \eta$$

and η is a small, positive real number. Then

$$\int_0^1 \int_0^1 L^*(x,y)\,dF\,dG = \gamma \int_0^1 \int_0^1 L(x,y)\,dF\,dG$$

$$\leq m < 1 \quad \text{for all } F,G \varepsilon D.$$

From McKinsey's fundamental theorem of continuous games, [7],

$$\max_F \min_G \int_0^1 \int_0^1 K(x,y)\,dF\,dG = \min_G \max_F \int_0^1 \int_0^1 K(x,y)\,dF\,dG.$$

Define the recurrence relation

$$u_o = \max_F \min_G \int_0^1 \int_0^1 K(x,y)\,dF\,dG$$

$$\cdots$$

Now, consider the approximation

$$u_1 = \underset{F}{\text{Max}}\ \underset{G}{\text{Min}} \{\int_0^1 \int_0^1 K(x,y)dFdG + [1 - \int_0^1 \int_0^1 L^*(x,y)dFdG]u_o\}$$

$$\simeq \underset{F}{\text{Max}}\ \underset{G}{\text{Min}} \int_0^1 \int_0^1 K(x,y)dFdG + [1-d]u_o$$

$$\simeq u_o + [1-d]u_o$$

$$u_1 - u_o \simeq [1 - d]u_o$$

$$u_2 \simeq \underset{F}{\text{Max}}\ \underset{G}{\text{Min}} \int_0^1 \int_0^1 K(x,y)dFdG + [1 - d]\ u_1$$

$$u_2 - u_1 \simeq [1 - d]\ (u_1 - u_o)$$

$$\simeq [1 - d]^2\ u_o$$

$$\cdots$$

$$u_n \simeq \underset{F}{\text{Max}}\ \underset{G}{\text{Min}} \int_0^1 \int_0^1 K(x,y)dFdG + [1 - d]\ u_{n-1}$$

$$u_{n+1} \simeq \underset{F}{\text{Max}}\ \underset{G}{\text{Min}} \int_0^1 \int_0^1 K(x,y)dFdG + [1 - d]\ u_n$$

$$u_{n+1} - u_n \simeq [1 - d]\ (u_n - u_{n-1})$$

$$\simeq [1-d]^{n+1}\ u_o\ .$$

The condition $0 < 1 - \int_0^1 \int_0^1 L^*(x,y)dFdG \leq 1 - d < 1$ yields geometric convergence of the series

and

$$\operatorname*{Max}_{F}\ \operatorname*{Min}_{G}\ \frac{\int_0^1\int_0^1 K(x,y)\,dF\,dG}{\int_0^1\int_0^1 L(x,y)\,dF\,dG} = \operatorname*{Min}_{G}\ \operatorname*{Max}_{F}\ \frac{\int_0^1\int_0^1 K(x,y)\,dF\,dG}{\int_0^1\int_0^1 L(x,y)\,dF\,dG},$$

which was the original hypothesis to be proved.

The preceding proof is an extension of an approximation method developed by Bellman [2]. Bellman proved a similar result for non-zero sum discrete games and discussed the possibility of using the ratio of individual returns as a criterion function for the evaluation of the optimal strategies for the game. The present extension shows the same method can be used to analyze no-zero sum continuous games.

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13. ABSTRACT

An analysis is made of the allocation problem associated with the conduct of ambush operations to interdict infiltration routes in a guerrilla-counterguerrilla environment. A multi-stage two-person non-zero sum game is used to model that allocation problem. It is shown that Lanchester's equations can be used to develop a criterion function, related to the casualty ratio, which demonstrates the minimax property. The game is then solved to determine the optimal allocations for both the guerrilla and the counterguerrilla and the value of the game for two different forms of the criterion function. The two results are compared and the usefulness of the casualty ratio as a measure of effectiveness is discussed.

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